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A new extension of a Hardy-Hilbert-type inequality

Qiliang Huang*

*Correspondence:
qlhuang@yeah.net
Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, P.R. China

Abstract

By introducing independent parameters, and applying weight coefficients and the technique of real analysis, we give a new extension of a Hardy-Hilbert-type inequality with a best possible constant factor. Furthermore, the equivalent forms, the operator expressions, and the reverses are considered.

MSC: 26D15; 47A07

Keywords: Hardy-Hilbert-type inequality; parameter; weight coefficient; equivalent form; reverse

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the Hardy-Hilbert inequality as follows (*cf.* [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. We also have the following Hardy-Hilbert-type inequality (*cf.* [2]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{2}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^2$ is still the best possible. In 2008, by introducing some parameters, Yang gave an extension of inequality (2) (*cf.* [3]): If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} n^{p(1-\lambda_1)-1} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q < \infty$, then the following inequality holds:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2\left(\sum_{n=1}^{\infty} n^{p(1-\lambda_1)-1}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right)^{1/q}, \tag{3}$$

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where the constant factor $[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2$ is the best possible. There are lots of improvements, generalizations, and applications of inequality (2) ([3–11]). For more details, Yang gives a summary of introducing independent parameters ([12, 13]).

In this article, by introducing independent parameters, and applying weight coefficients and the technique of real analysis, we give a new extension of (2) with a best possible constant factor. Furthermore, the equivalent forms, the operator expressions, and the reverses are considered.

2 Some lemmas

We agree on the following assumptions in this paper: $p \neq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $0 < \lambda_i \leq 1$ ($i = 1, 2$), $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(\lambda_2) = k_\lambda(\lambda_1) = [\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2$, $\{\mu_m\}_{m=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ are positive sequences, $U_m = \sum_{i=1}^m \mu_i$, $V_n = \sum_{i=1}^n \nu_i$, and $a_n, b_n \geq 0$ ($m, n \in \mathbf{N} = \{1, 2, \ldots\}$),

$$0 < \sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p < \infty, \qquad 0 < \sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q < \infty.$$

Lemma 1 *Define the weight coefficients as follows*:

$$\omega(\lambda_2, m) := \sum_{n=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} \nu_n, \quad m \in \mathbf{N}, \tag{4}$$

$$\varpi(\lambda_1, n) := \sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{V_n^{\lambda_2}}{U_m^{1-\lambda_1}} \mu_m, \quad n \in \mathbf{N}. \tag{5}$$

We have the following inequalities:

$$\omega(\lambda_2, m) < k_\lambda(\lambda_1) \quad (m \in \mathbf{N}; 0 < \lambda_2 \leq 1, \lambda_1 > 0), \tag{6}$$

$$\varpi(\lambda_1, n) < k_\lambda(\lambda_1) \quad (n \in \mathbf{N}; 0 < \lambda_1 \leq 1, \lambda_2 > 0). \tag{7}$$

Proof Putting $\mu(t) := \mu_m$, $t \in (m-1, m]$ ($m = 1, 2, \ldots$), $\nu(t) := \nu_n$, $t \in (n-1, n]$ ($n = 1, 2, \ldots$),

$$U(x) := \int_0^x \mu(t)\,dt \quad (x \geq 0), \qquad V(y) := \int_0^y \nu(t)\,dt \quad (y \geq 0).$$

Then we have $U(m) = U_m$, $V(n) = V_n$ ($m, n \in \mathbf{N}$). $U'(x) = \mu(x) = \mu_m$ when $x \in (m-1, m]$; $V'(y) = \nu(y) = \nu_n$ when $y \in (n-1, n]$. Since the function $V(y)$ ($y > 0$) is strictly increasing and $f(x) = \frac{\ln(m/x)}{m^\lambda - x^\lambda}$ ($x > 0$) is strictly decreasing (*cf.* [4], Example 2.2.1), in view of $1 - \lambda_2 \geq 0$, we have

$$\begin{aligned} \omega(\lambda_2, m) &= \sum_{n=1}^\infty \int_{n-1}^n \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} V'(t)\,dt \\ &< \sum_{n=1}^\infty \int_{n-1}^n \frac{\ln(U_m/V(t))}{U_m^\lambda - V^\lambda(t)} \frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(t)} V'(t)\,dt. \end{aligned}$$

Putting $u = \frac{V^\lambda(t)}{U_m^\lambda}$ in the above integral, and in view of the fact that (*cf.* [2])

$$\int_0^\infty \frac{\ln u}{u-1} u^{a-1}\,du = \left[\frac{\pi}{\sin(a\pi)}\right]^2 \quad (0 < a < 1),$$

it follows that

$$\begin{aligned}
\omega(\lambda_2, m) &< \frac{1}{\lambda^2} \sum_{n=1}^{\infty} \int_{\frac{V^\lambda(n-1)}{U_m^\lambda}}^{\frac{V^\lambda(n)}{U_m^\lambda}} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du \\
&= \frac{1}{\lambda^2} \int_0^{\frac{V^\lambda(\infty)}{U_m^\lambda}} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du \le \frac{1}{\lambda^2} \int_0^{\infty} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du \\
&= \left[\frac{\pi}{\lambda \sin(\pi\lambda_2/\lambda)}\right]^2 = \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 = k_\lambda(\lambda_1).
\end{aligned}$$

Hence we prove that (6) is valid. In the same way, we can prove that (7) is valid too. □

Lemma 2 *Suppose that* $\{\mu_m\}_{m=1}^\infty$ *and* $\{\nu_n\}_{n=1}^\infty$ *are decreasing sequences, and* $U(\infty) = V(\infty) = \infty$, *then we have the following inequalities*:

$$k_\lambda(\lambda_1)\bigl(1 - \theta_1(\lambda_2, m)\bigr) < \omega(\lambda_2, m) \quad (m \in \mathbf{N}; 0 < \lambda_2 \le 1, \lambda_1 > 0), \tag{8}$$

$$k_\lambda(\lambda_1)\bigl(1 - \theta_2(\lambda_1, n)\bigr) < \varpi(\lambda_1, n) \quad (n \in \mathbf{N}; 0 < \lambda_1 \le 1, \lambda_2 > 0), \tag{9}$$

where $\theta_1(\lambda_2, m) = O(\frac{1}{U_m^{\lambda_2/2}}) \in (0,1)$ *and* $\theta_2(\lambda_1, n) = O(\frac{1}{V_n^{\lambda_1/2}}) \in (0,1)$. *Moreover, we get*

$$\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1 + o_1(1)\bigr) \quad \bigl(\varepsilon \to 0^+\bigr), \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1 + o_2(1)\bigr) \quad \bigl(\varepsilon \to 0^+\bigr). \tag{11}$$

Proof By the decreasing property of $\{\nu_n\}_{n=1}^\infty$, and in view of $1 - \lambda_2 \ge 0$, $V(\infty) = \infty$, we find

$$\begin{aligned}
\omega(\lambda_2, m) &\ge \sum_{n=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} \nu_{n+1} \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} V'(t)\, dt \\
&> \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\ln(U_m/V(t))}{U_m^\lambda - V^\lambda(t)} \frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(t)} V'(t)\, dt \\
&= \frac{1}{\lambda^2} \sum_{n=1}^{\infty} \int_{\frac{V^\lambda(n)}{U_m^\lambda}}^{\frac{V^\lambda(n+1)}{U_m^\lambda}} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du = \frac{1}{\lambda^2} \int_{\frac{V^\lambda(1)}{U_m^\lambda}}^{\infty} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du \\
&= k_\lambda(\lambda_1) - \frac{1}{\lambda^2} \int_0^{\frac{\nu_1^\lambda}{U_m^\lambda}} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du = k_\lambda(\lambda_1)\bigl(1 - \theta_1(\lambda_2, m)\bigr),
\end{aligned}$$

where

$$\theta_1(\lambda_2, m) := \frac{1}{\lambda^2 k_\lambda(\lambda_1)} \int_0^{\frac{\nu_1^\lambda}{U_m^\lambda}} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\, du \in (0,1).$$

In virtue of

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\int_0^{\nu_1^\lambda/x^\lambda} \frac{\ln u}{u-1} u^{\frac{\lambda_2}{\lambda}-1}\,du}{x^{-\lambda_2/2}} \\
&\quad= \lim_{x\to\infty} \frac{2\lambda^2\nu_1^{\lambda_2}}{\lambda_2}\left(\frac{\nu_1^\lambda}{x^\lambda}-1\right)^{-1}\left(\frac{1}{x^{\lambda_2/2}}\ln\frac{\nu_1}{x}\right) = 0,
\end{aligned}$$

it is obvious that $\theta_1(\lambda_2, m) = O(\frac{1}{U_m^{\lambda_2/2}})$. Hence (8) is valid. In the same way, we can prove that (9) is valid too. Moreover, we have

$$\begin{aligned}
\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}} &= \frac{1}{\mu_1^\varepsilon} + \sum_{m=2}^\infty \int_{m-1}^m \frac{U'(t)}{U_m^{1+\varepsilon}}\,dt \\
&\le \frac{1}{\mu_1^\varepsilon} + \sum_{m=2}^\infty \int_{m-1}^m \frac{U'(t)}{U^{1+\varepsilon}(t)}\,dt \\
&= \frac{1}{\mu_1^\varepsilon} + \sum_{m=2}^\infty \int_{U(m-1)}^{U(m)} \frac{1}{u^{1+\varepsilon}}\,du = \frac{1}{\mu_1^\varepsilon} + \int_{\mu_1}^\infty \frac{1}{u^{1+\varepsilon}}\,du \\
&= \frac{1}{\varepsilon}\left[1 + \left(\frac{1}{\mu_1^\varepsilon} + \frac{\varepsilon}{\mu_1^\varepsilon} - 1\right)\right], \\
\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}} &\ge \sum_{m=1}^\infty \int_m^{m+1} \frac{\mu_{m+1}}{U_m^{1+\varepsilon}}\,dt \\
&= \sum_{m=1}^\infty \int_m^{m+1} \frac{U'(t)}{U_m^{1+\varepsilon}}\,dt > \sum_{m=1}^\infty \int_m^{m+1} \frac{U'(t)}{U^{1+\varepsilon}(t)}\,dt \\
&= \sum_{m=1}^\infty \int_{U(m)}^{U(m+1)} \frac{1}{u^{1+\varepsilon}}\,du = \int_{\mu_1}^\infty \frac{1}{u^{1+\varepsilon}}\,du \\
&= \frac{1}{\varepsilon}\left[1 + \left(\frac{1}{\mu_1^\varepsilon} - 1\right)\right].
\end{aligned}$$

Then we have (10). In the same way, we have (11). □

Remark 1 Taking $\varepsilon = a > 0$, we write by (10) and (11) that

$$\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+a}} = O_1(1), \qquad \sum_{n=1}^\infty \frac{\nu_n}{V_n^{1+a}} = O_2(1).$$

3 Equivalent forms and operator expressions

Theorem 1 *Suppose that $p > 1$, then we have the following equivalent inequalities*:

$$\begin{aligned}
I &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m b_n \\
&< \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \left[\sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p\right]^{1/p} \left[\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q\right]^{1/q},
\end{aligned} \tag{12}$$

$$\begin{aligned} J &:= \left\{ \sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\lambda_2}} \left(\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right)^p \right\}^{\frac{1}{p}} \\ &< \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)} \right]^2 \left(\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right)^{1/p}. \end{aligned} \tag{13}$$

Proof By Hölder's inequality with weight (*cf.* [14]), we find

$$\begin{aligned} &\left(\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right)^p \\ &\quad = \left\{ \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \left[\frac{U_m^{(1-\lambda_1)/q} \nu_n^{1/p}}{V_n^{(1-\lambda_2)/p} \mu_m^{1/q}} a_m \right] \left[\frac{V_n^{(1-\lambda_2)/p} \mu_m^{1/q}}{U_m^{(1-\lambda_1)/q} \nu_n^{1/p}} \right] \right\}^p \\ &\quad \le \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \frac{U_m^{(1-\lambda_1)p/q} \nu_n}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p \left[\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \frac{V_n^{(1-\lambda_2)(q-1)} \mu_m}{U_m^{1-\lambda_1} \nu_n^{q-1}} \right]^{p-1} \\ &\quad = \left(\varpi(\lambda_1, n) \right)^{p-1} \frac{V_n^{1-p\lambda_2}}{\nu_n} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \frac{U_m^{(1-\lambda_1)p/q} \nu_n}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p. \end{aligned} \tag{14}$$

By (7), it follows that

$$\begin{aligned} J &< \left(k_\lambda(\lambda_1) \right)^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \frac{U_m^{(1-\lambda_1)p/q} \nu_n}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p \right]^{\frac{1}{p}} \\ &= \left(k_\lambda(\lambda_1) \right)^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} \frac{U_m^{(1-\lambda_1)(p-1)} \nu_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \\ &= \left(k_\lambda(\lambda_1) \right)^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}}. \end{aligned} \tag{15}$$

Combining (8) and (15), we have (13).

Using Hölder's inequality again, we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[\frac{\nu_n^{1/p}}{V_n^{\frac{1}{p}-\lambda_2}} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right] \left[\frac{V_n^{\frac{1}{p}-\lambda_2}}{\nu_n^{1/p}} b_n \right] \\ &\le J \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{16}$$

and then we have (12) by using (13). On the other hand, assuming that (12) is valid, setting

$$b_n = \frac{\nu_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right]^{p-1}, \quad n \in \mathbf{N},$$

then we find $J = [\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q]^{1/p}$. By (15), it follows that $J < \infty$. If $J = 0$, then (13) is trivially valid. If $0 < J < \infty$, then we have

$$
\begin{aligned}
0 &< \sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q = J^p = I \\
&< k_\lambda(\lambda_1) \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{\frac{1}{q}} < \infty, \\
J &= \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{1/p} < k_\lambda(\lambda_1) \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}}.
\end{aligned}
$$

Hence (13) is valid, which is equivalent to (12). □

Theorem 2 *Suppose that $p > 1$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are decreasing positive sequences, and $U(\infty) = V(\infty) = \infty$, then the constant factor $k_\lambda(\lambda_1) = [\frac{\pi}{\lambda \sin(\lambda_1 \pi/\lambda)}]^2$ is the best possible in* (12) *and* (13).

Proof For $0 < \varepsilon < p\lambda_1$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,1))$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (> 0), $\widetilde{a}_m = U_m^{\widetilde{\lambda}_1 - 1}\mu_m$, $\widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - \varepsilon - 1}\nu_n$. By (10), (11), and (9), in view of Remark 1, we find

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} \widetilde{a}_m^p = \sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1 + o_1(1)\bigr), \\
&\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} \widetilde{b}_n^q = \sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1 + o_2(1)\bigr), \\
&\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \widetilde{a}_m \widetilde{b}_n \\
&\quad = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}} \right] \frac{\nu_n}{V_n^{\varepsilon+1}} \\
&\quad = \sum_{n=1}^{\infty} \varpi(\widetilde{\lambda}_1, n) \frac{\nu_n}{V_n^{\varepsilon+1}} \geq k_\lambda(\widetilde{\lambda}_1) \sum_{n=1}^{\infty} \bigl(1 - \theta_2(\widetilde{\lambda}_1, n)\bigr) \frac{\nu_n}{V_n^{\varepsilon+1}} \\
&\quad = k_\lambda(\widetilde{\lambda}_1) \left[\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{\varepsilon+1}} - \sum_{n=1}^{\infty} O\left(\frac{\nu_n}{V_n^{\frac{1}{2}(\frac{\varepsilon}{q}+\varepsilon+\lambda_1)+1}} \right) \right] \\
&\quad = \frac{1}{\varepsilon} \left[\frac{\pi}{\lambda \sin(\pi \widetilde{\lambda}_1/\lambda)} \right]^2 \bigl[1 + o_2(1) - \varepsilon O(1)\bigr].
\end{aligned}
$$

If there exists a positive number $K \leq k_\lambda(\lambda_1)$, such that (12) is still valid when replacing $k_\lambda(\lambda_1)$ by K, then, in particular, we have

$$
\begin{aligned}
\varepsilon \widetilde{I} &= \varepsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} \widetilde{a}_m \widetilde{b}_n \\
&< \varepsilon K \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} \widetilde{a}_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} \widetilde{b}_n^q \right]^{\frac{1}{q}}.
\end{aligned}
$$

We obtain from the above results

$$\left[\frac{\pi}{\lambda\sin(\pi\widetilde{\lambda}_1/\lambda)}\right]^2\left[1+o_2(1)-\varepsilon O(1)\right]<K\bigl(1+o_1(1)\bigr)^{\frac{1}{p}}\bigl(1+o_2(1)\bigr)^{\frac{1}{q}},$$

and then it follows that $k_\lambda(\lambda_1)\le K$ (for $\varepsilon\to 0^+$). Hence $K=k_\lambda(\lambda_1)$ is the best value of (12).

We conform that the constant factor $k_\lambda(\lambda_1)$ in (13) is the best possible. Otherwise we can get a contradiction by (16): that the constant factor in (12) is not the best value. □

For $p>1$, setting

$$\varphi(m):=\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\qquad \psi(n):=\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}\quad(n,m\in\mathbf{N}),$$

then it follows that $[\psi(n)]^{1-p}=\frac{\nu_n}{V_n^{1-p\lambda_2}}$, and we define the real weighted normed function spaces as follows:

$$l_{p,\varphi}:=\left\{a=\{a_m\}_{m=1}^\infty;\|a\|_{p,\varphi}=\left\{\sum_{m=1}^\infty\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}|a_m|^p\right\}^{\frac{1}{p}}<\infty\right\},$$

$$l_{q,\psi}:=\left\{b=\{b_n\}_{n=1}^\infty;\|b\|_{q,\psi}=\left\{\sum_{n=1}^\infty\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}|b_n|^q\right\}^{\frac{1}{q}}<\infty\right\},$$

$$l_{p,\psi^{1-p}}:=\left\{c=\{c_n\}_{n=1}^\infty;\|c\|_{p,\psi^{1-p}}=\left\{\sum_{n=1}^\infty\frac{\nu_n}{V_n^{1-p\lambda_2}}|c_n|^p\right\}^{\frac{1}{p}}<\infty\right\}.$$

For $a=\{a_m\}_{m=1}^\infty\in l_{p,\varphi}$, putting $h_n:=\sum_{m=1}^\infty\frac{\ln(U_m/V_n)}{U_m^\lambda-V_n^\lambda}a_m$, $h=\{h_n\}_{n=1}^\infty$, then it follows by (13) that $\|h\|_{p,\psi^{1-p}}<k_\lambda(\lambda_1)\|a\|_{p,\varphi}$, and $h\in l_{p,\psi^{1-p}}$.

Definition 1 Define a Hardy-Hilbert-type operator $T:l_{p,\varphi}\to l_{p,\psi^{1-p}}$ as follows: For $a_m\ge 0$, $a=\{a_m\}_{m=1}^\infty\in l_{p,\varphi}$, there exists a unique representation $Ta=h\in l_{p,\psi^{1-p}}$. We define the following formal inner product of Ta and $b=\{b_n\}_{n=1}^\infty\in l_{q,\psi}$ $(b_n\ge 0)$ as follows:

$$(Ta,b):=\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{\ln(U_m/V_n)}{U_m^\lambda-V_n^\lambda}a_mb_n. \tag{17}$$

Hence (12) and (13) may be rewritten in terms of the following operator expressions:

$$(Ta,b)<k_\lambda(\lambda_1)\|a\|_{p,\varphi}\|b\|_{q,\psi}, \tag{18}$$

$$\|Ta\|_{p,\psi^{1-p}}<k_\lambda(\lambda_1)\|a\|_{p,\varphi}. \tag{19}$$

It follows that the operator T is bounded with

$$\|T\|:=\sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}}\le k_\lambda(\lambda_1).$$

Since the constant factor $k_\lambda(\lambda_1)$ in (19) is the best possible, we have

$$\|T\| = k_\lambda(\lambda_1) = \left[\frac{\pi}{\lambda \sin(\lambda_1\pi/\lambda)}\right]^2. \tag{20}$$

4 Some reverses

We set $\widetilde{\varphi}(m) := (1-\theta_1(\lambda_2,m))\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}$, $\widetilde{\psi}(n) := (1-\theta_2(\lambda_1,m))\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}$ $(n,m \in \mathbf{N})$. For $0 < p < 1$ or $p < 0$, we still use the formal symbol of the norm in this part for convenience.

Theorem 3 *Suppose that* $0 < p < 1$, $\{\mu_m\}_{m=1}^\infty$ *and* $\{\nu_n\}_{n=1}^\infty$ *are decreasing positive sequences, and* $U(\infty) = V(\infty) = \infty$, *then we have the following equivalent inequalities*:

$$I = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m b_n > \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \|a\|_{p,\widetilde{\varphi}} \|b\|_{q,\psi}, \tag{21}$$

$$\begin{aligned} J &= \left\{\sum_{n=1}^\infty \frac{\nu_n}{V_n^{1-p\lambda_2}} \left(\sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m\right)^p\right\}^{\frac{1}{p}} \\ &> \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \|a\|_{p,\widetilde{\varphi}} \end{aligned} \tag{22}$$

where the constant factor $[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2$ *is the best possible.*

Proof By the reverse Hölder inequality with weight (*cf.* [14]) and (7), we obtain the reverse forms of (14) and (15). It follows that (22) is valid by (8). Using the reverse Hölder inequality (*cf.* [14]), we find

$$I = \sum_{n=1}^\infty \left[\frac{\nu_n^{1/p}}{V_n^{\frac{1}{p}-\lambda_2}} \sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m\right]\left[\frac{V_n^{\frac{1}{p}-\lambda_2}}{\nu_n^{1/p}} b_n\right] \ge J\|b\|_{q,\psi}. \tag{23}$$

Hence (21) is valid by using (22). Setting

$$b_n = \frac{\nu_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m\right]^{p-1}, \quad n \in \mathbf{N},$$

then we have $J = [\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q]^{1/p}$. Assume that (21) is valid. By the reverse of (15), it follows that $J > 0$. If $J = \infty$, then (22) is trivially valid. If $0 < J < \infty$, then we find

$$\begin{aligned} &\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q = J^p = I > k_\lambda(\lambda_1)\|a\|_{p,\widetilde{\varphi}} \left[\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q\right]^{\frac{1}{q}}, \\ &J = \left[\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q\right]^{1/p} > k_\lambda(\lambda_1)\|a\|_{p,\widetilde{\varphi}}. \end{aligned}$$

Hence (22) is valid, which is equivalent to (21).

For $0 < \varepsilon < p\lambda_1$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,1))$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (> 0), $\widetilde{a}_m = U_m^{\widetilde{\lambda}_1 - 1}\mu_m$, $\widetilde{b}_n = V_n^{\widetilde{\lambda}_2-\varepsilon-1}\nu_n$. By (10), (11), and (7), in view of Remark 1, we find

$$\begin{aligned}
&\sum_{m=1}^{\infty}\bigl(1-\theta_1(\lambda_2,m)\bigr)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}\widetilde{a}_m^p \\
&\quad= \sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{U_m^{\lambda_2/2}}\right)\right)\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&\quad= \sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} - \sum_{m=1}^{\infty}O\left(\frac{\mu_m}{U_m^{1+\varepsilon+(\lambda_2/2)}}\right) = \frac{1}{\varepsilon}\bigl(1+o_1(1)-\varepsilon O(1)\bigr), \\
&\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}\widetilde{b}_n^q = \sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1+o_2(1)\bigr), \\
&\widetilde{I} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\widetilde{a}_m\widetilde{b}_n = \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right]\frac{\nu_n}{V_n^{\varepsilon+1}} \\
&\quad= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\nu_n}{V_n^{\varepsilon+1}} < k_\lambda(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{\varepsilon+1}} \\
&\quad= \frac{1}{\varepsilon}\left[\frac{\pi}{\lambda\sin(\pi\widetilde{\lambda}_1/\lambda)}\right]^2\bigl(1+o_2(1)\bigr).
\end{aligned}$$

If there exists a positive number $K \geq k_\lambda(\lambda_1)$, such that (21) is still valid when replacing $k_\lambda(\lambda_1)$ by K, then in particular, we have

$$\begin{aligned}
\varepsilon\widetilde{I} &= \varepsilon\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\widetilde{a}_m\widetilde{b}_n \\
&> \varepsilon K\left[\sum_{m=1}^{\infty}\bigl(1-\theta_1(\lambda_2,m)\bigr)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}\widetilde{b}_n^q\right]^{\frac{1}{q}}.
\end{aligned}$$

We obtain from the above results that

$$\left[\frac{\pi}{\lambda\sin(\pi\widetilde{\lambda}_1/\lambda)}\right]^2\bigl(1+o_2(1)\bigr) > K\bigl(1+o_1(1)-\varepsilon O(1)\bigr)^{\frac{1}{p}}\bigl(1+o_2(1)\bigr)^{\frac{1}{q}},$$

and then $k_\lambda(\lambda_1) \geq K$ (for $\varepsilon \to 0^+$). Hence $K = k_\lambda(\lambda_1)$ is the best value of (21).

We conform that the constant factor $k_\lambda(\lambda_1)$ in (22) is the best possible. Otherwise we can get a contradiction by (23): that the constant factor in (21) is not the best value. □

Theorem 4 *Suppose that* $p < 0$, $\{\mu_m\}_{m=1}^{\infty}$ *and* $\{\nu_n\}_{n=1}^{\infty}$ *are decreasing positive sequences, and* $U(\infty) = V(\infty) = \infty$, *then we have the following equivalent inequalities*:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}a_m b_n > \left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2\|a\|_{p,\varphi}\|b\|_{q,\widetilde{\psi}}, \tag{24}$$

$$J_1 = \left\{ \sum_{n=1}^{\infty} \frac{(1-\theta_2(\lambda_1, n))^{1-p} \nu_n}{V_n^{1-p\lambda_2}} \left(\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right)^p \right\}^{\frac{1}{p}}$$
$$> \left[\frac{\pi}{\lambda \sin(\pi \lambda_1/\lambda)} \right]^2 \|a\|_{p,\varphi}, \tag{25}$$

where the constant factor $[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}]^2$ *is the best possible.*

Proof Using the same way of obtaining (14) and (15), by the reverse Hölder inequality with weight and (9), we have

$$J_1 > \big(k_\lambda(\lambda_1)\big)^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}}, \tag{26}$$

then we obtain (25) by (6). Using the reverse Hölder inequality, we have

$$I = \sum_{n=1}^{\infty} \left[\frac{(1-\theta_2(\lambda_1, n))^{-\frac{1}{q}} \nu_n^{1/p}}{V_n^{\frac{1}{p}-\lambda_2}} \sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right]$$
$$\times \left[\big(1-\theta_2(\lambda_1, n)\big)^{\frac{1}{q}} \frac{V_n^{\frac{1}{p}-\lambda_2}}{\nu_n^{1/p}} b_n \right]$$
$$\geq J_1 \|b\|_{q,\widetilde{\psi}}. \tag{27}$$

Hence (24) is valid by (25). Assuming that (24) is valid, setting

$$b_n = \frac{(1-\theta_2(\lambda_1, n))^{1-p} \nu_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^{\infty} \frac{\ln(U_m/V_n)}{U_m^{\lambda} - V_n^{\lambda}} a_m \right]^{p-1}, \quad n \in \mathbf{N},$$

we find

$$J_1 = \left[\sum_{n=1}^{\infty} \big(1-\theta_2(\lambda_1, n)\big) \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{1/p}.$$

It follows that $J_1 > 0$ by (26). If $J_1 = \infty$, then (25) is trivially valid. If $0 < J_1 < \infty$, then we find

$$\sum_{n=1}^{\infty} \big(1-\theta_2(\lambda_1, n)\big) \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q = J_1^p = I$$
$$> k_\lambda(\lambda_1) \|a\|_{p,\varphi} \left[\sum_{n=1}^{\infty} \big(1-\theta_2(\lambda_1, n)\big) \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{\frac{1}{q}},$$
$$J_1 = \left[\sum_{n=1}^{\infty} \big(1-\theta_2(\lambda_1, n)\big) \frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}} b_n^q \right]^{1/p} > k_\lambda(\lambda_1) \|a\|_{p,\varphi}.$$

Hence (25) is valid, which is equivalent to (24).

For $0 < \varepsilon < q\lambda_2$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ (> 0), $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ ($\in (0,1)$), $\widetilde{a}_m = U_m^{\widetilde{\lambda}_1-\varepsilon-1}\mu_m$, $\widetilde{b}_n = V_n^{\widetilde{\lambda}_2-1}\nu_n$. By (10), (11), and (6), in view of Remark 1, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}\widetilde{a}_m^p = \sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}} = \frac{1}{\varepsilon}\bigl(1+o_1(1)\bigr), \\
&\sum_{n=1}^{\infty}\bigl(1-\theta_2(\lambda_1,n)\bigr)\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}\widetilde{b}_n^q \\
&\quad = \sum_{n=1}^{\infty}\biggl(1-O\biggl(\frac{1}{V_n^{\lambda_1/2}}\biggr)\biggr)\frac{\nu_n}{V_n^{1+\varepsilon}} \\
&\quad = \frac{1}{\varepsilon}\bigl(1+o_2(1)-\varepsilon O(1)\bigr), \\
&\widetilde{I} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\widetilde{a}_m\widetilde{b}_n \\
&\quad = \sum_{m=1}^{\infty}\Biggl[\sum_{n=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\frac{U_m^{\widetilde{\lambda}_1}\nu_n}{V_n^{1-\widetilde{\lambda}_2}}\Biggr]\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&\quad = \sum_{m=1}^{\infty}\varpi(\widetilde{\lambda}_2,m)\frac{\mu_m}{U_m^{1+\varepsilon}} < k_\lambda(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&\quad = \frac{1}{\varepsilon}\biggl[\frac{\pi}{\lambda\sin(\pi\widetilde{\lambda}_1/\lambda)}\biggr]^2\bigl(1+o_1(1)\bigr).
\end{aligned}$$

If there exists a positive number $K \geq k_\lambda(\lambda_1)$, such that (24) is still valid as we replace $k_\lambda(\lambda_1)$ by K, then, in particular, we have

$$\begin{aligned}
\varepsilon\widetilde{I} &= \varepsilon\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m^{\lambda}-V_n^{\lambda}}\widetilde{a}_m\widetilde{b}_n \\
&> \varepsilon K\Biggl[\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}\widetilde{a}_m^p\Biggr]^{\frac{1}{p}}\Biggl[\sum_{n=1}^{\infty}\bigl(1-\theta_2(\lambda_1,n)\bigr)\frac{V_n^{q(1-\lambda_2)-1}}{\nu_n^{q-1}}\widetilde{b}_n^q\Biggr]^{\frac{1}{q}}.
\end{aligned}$$

From the above results, we have

$$\biggl[\frac{\pi}{\lambda\sin(\pi\widetilde{\lambda}_1/\lambda)}\biggr]^2\bigl(1+o_1(1)\bigr) > K\bigl(1+o_1(1)\bigr)^{\frac{1}{p}}\bigl(1+o_2(1)-\varepsilon O(1)\bigr)^{\frac{1}{q}}.$$

It follows that $k_\lambda(\lambda_1) \geq K$ (for $\varepsilon \to 0^+$). Hence $K = k_\lambda(\lambda_1)$ is the best value of (24). We conform that the constant factor $k_\lambda(\lambda_1)$ in (25) is the best possible. Otherwise we can get a contradiction by (27): that the constant factor in (24) is not the best value. □

Remark 2 For $\mu_i = \nu_i = 1$ ($i = 1,2,\ldots$), (12) reduces to (3); for $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, it follows by (12) that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m-V_n}a_m b_n < \biggl[\frac{\pi}{\sin(\pi/p)}\biggr]^2\Biggl[\sum_{m=1}^{\infty}\frac{1}{\mu_m^{p-1}}a_m^p\Biggr]^{1/p}\Biggl[\sum_{n=1}^{\infty}\frac{1}{\nu_n^{q-1}}b_n^q\Biggr]^{1/q}; \qquad (28)$$

for $\lambda = 1$, $\lambda_1 = \frac{1}{p}$, $\lambda_2 = \frac{1}{q}$, (12) reduces to the dual form of (28) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(U_m/V_n)}{U_m - V_n}a_m b_n < \left[\frac{\pi}{\sin(\pi/p)}\right]^2\left[\sum_{m=1}^{\infty}\frac{U_m^{p-2}}{\mu_m^{p-1}}a_m^p\right]^{1/p}\left[\sum_{n=1}^{\infty}\frac{V_n^{q-2}}{\nu_n^{q-1}}b_n^q\right]^{1/q}. \tag{29}$$

Competing interests
The author declares to have no competing interests.

Author's contributions
QH carried out the mathematical studies, sequenced alignment, drafted the manuscript, and performed the numerical analysis. The author read and approved the final manuscript.

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References
1. Hardy, GH: Note on a theorem of Hilbert concerning series of positive terms. Proc. Lond. Math. Soc. (2) **23**, xlv-xlvi (1925)
2. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
3. Yang, BC: An extension of the Hilbert-type inequality and its reverse. J. Math. Inequal. **2**(1), 139-149 (2008)
4. Yang, BC: Generalization of Hilbert's type inequality with best constant factor and its applications. J. Math. Res. Expo. **25**(2), 341-346 (2005) (in Chinese)
5. Yang, BC: On a more accurate Hardy-Hilbert's type inequality and its applications. Acta Math. Sin. New Ser. **49**(2), 363-368 (2006)
6. Yang, BC: On the norm of a Hilbert's type linear operator and applications. J. Math. Anal. Appl. **325**, 529-541 (2007)
7. Yang, BC: On a new Hardy-Hilbert's type inequality with a parameter. Int. J. Math. Anal. **1**, 123-131 (2007)
8. Wang, WH, Yang, BC: A strengthened Hardy-Hilbert's type inequality. Aust. J. Math. Anal. Appl. **2**, 1-7 (2006)
9. Zhong, WY, Yang, BC: A reverse Hilbert's type integral inequality with some parameters and the equivalent forms. Pure Appl. Math. **24**(2), 401-407 (2008)
10. He, B: On a bilateral Hilbert-type inequality with a homogeneous kernel of 0-degree. Kyungpook Math. J. **50**, 307-314 (2010)
11. Xin, DM, Yang, BC: A basic Hilbert-type inequality. J. Math. **30**(3), 554-560 (2010)
12. Yang, BC: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijin (2009) (in Chinese)
13. Yang, BC: Discrete Hilbert-Type Inequalities. Bentham Science Publishers Ltd, Sharjah (2011)
14. Kuang, J: Applied Inequalities. Shangdong Science Technic Press, Jinan (2010) (in Chinese)